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COMMON FIXED POINTS OF RELATIVELY NONEXPANSIVE MAPPINGS BY ITERATION

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Abstract

Let us consider two nonempty closed convex subsets A, B of a strictly convex space and $f_i : A \cup B \to A \cup B$, i = 1, 2, . . . k be a reltively nonexpansive mappings. ie. $f_i(A) \subseteq A$ and $f_i(B) \subseteq B$ and $\|f_i x - f_i y\| \leq \|x - y\|$, for all $x \in A$ and $y \in B$. In this paper, we provide the strong convergence of some iteration of the mappings $\{f_i\}^k_1$ to a common fixed point of $\{f_i\}^k_1$ in strictly convex space setting, which generalizes a result of Kuhfittig [7].

Key words: *Relatively nonexpansive mappings, fixed points.*

AMS Subject classification: 54H25,47H10.

I. INTRODUCTION

We know that the behaviour of the iterated sequences play an role in fixed point theory. It is well known fact that if an iterated sequence of a continuous mapping T converges, then the limit of it must be a fixed point of T . Also, Banach contraction principle states that every contraction mapping $T : A \to A$, where A is a complete subspace of a metric space X, has unique fixed point in A and every iterated sequence of T starting from any $x \in A$ converges to the unique fixed point of T . But the behaviour of the iterated sequences of nonexpansive mappings are completely different from the iterated sequences of contractive type mappings.

Consider a nonexpansive mapping $T : A \to A$, where A is a nonempty closed convex subset of a normed linear space X. In [1], Krasnoselskii proved that in uniformly convex Banach space X, the sequence of successive approximation of the averaged mapping $F : A \to A$ given by $F(x) := (x + Tx)/2$, for all $x \in A$, converges to a fixed point of the nonexpansive mappings T . A complete proof of Krasnoselskii's results in English can be found in [2]. Later, in [3], Edelstein extended Krasnoselskii's result to strictly convex space setting.

In [4], the authors introduced a class of mappings called relatively nonexpansive defined as follows, which generalizes the notion of nonexpansive mappings.

DEFINITION 1. Let A, B be nonempty subsets of a normed linear space X and $T : A \cup B \to A \cup B$ be a mapping. Then T is said to be a relatively nonexpansive mapping if and only if

1. $T(A) \subseteq A$ and $T(B) \subseteq B$,
2. $\|Tx - Ty\| \leq \|x - y\|$, for all $x \in A$, $y \in B$.

Define that $dist(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}$ and for any given pair of subsets A, B of a normed linear space X, define $A_0 = \{x \in A; \|x - y\| = dist(A, B)$, for some $y \in B\}$. In [5], the authors provided sufficient conditions

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which ensure the non emptiness of the set A_0. In [6], the authors proved that A_0 is contained in the boundary of the set A.

In [4], the authors introduced and used the geometric notion called proximal normal structure to prove the existence of the best proximity point. In [8], the authors generalized the results in [4]. In [7], the main result is as follows.

THEOREM 1.1. Let C be a convex compact subset of a strictly convex Banach space X and $\{T_i : i = 1, 2, \ldots, k\}$ a family of non-expansive self mappings of C with a nonempty set of common fixed points.
Then for an arbitrary starting point $x \in C$, the sequence $\{U_k^n x\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$.

In this article, we generalized the above theorem of [7].

II. PRELIMINARIES

In this section, we introduce basic definition and results which we used in our main result. We generalized the iteration of nonexpansive given in [7]

REMARK 2.1. Let A, B be two nonempty convex subsets of a Banach space X. Let $f_i : A\cup B \to A\cup B$, $i = 1, 2, \ldots k$, be a reltively nonexpansive mapping. Fix $F_0 = I$. For $0 < \alpha < 1$.

$$\begin{aligned} \text{Let } F_1 &= (1-\alpha)I + \alpha f_1F_0 \\ F_2 &= (1-\alpha)I + \alpha f_2F_1 \\ &\vdots \\ F_k &= (1-\alpha)I + \alpha f_kF_{k-1}. \end{aligned}$$

$$x_{n+1} = (1-\alpha)\, x_n + \alpha\, f_k\, F_{k-1}\, x_n \qquad (1)$$

$$\text{Put } k = 1,\ x_{n+1} = (1-\alpha)\, x_n + \alpha\, f_1\, F_{0\, xn} \qquad (2)$$
$$= (1-\alpha)\, x_n + \alpha\, f_1\, x_n$$

Let us state an convergence result, which plays a vital role in our main result.

THEOREM 2.1. [8]Let A, B be nonempty closed convex subsets of a strictly convex Banach space X such that A_0 is nonempty. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Suppose T (A) is contained in a compact subset A_1 of A. Then the Krasnoselskii's iteration $\{F^n(x)\}$, where $F: A\cup B \to A\cup B$ given by
F (x) = 1\2(T x+x), converges to a fixed point of T .

III. MAIN RESULT

Our main result is as follows.

THEOREM 3.1. Let A, B be two nonempty convex, compact subsets of a strictly convex Banach space X with A_0 is nonempty. Let $f_i : A \cup B \to A \cup B$, $i = 1, 2, 3 \ldots, k$ be mappings with a non empty set of fixed points $\|f_i(x)$

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$-f_i(y)\| \leq \|x - y\|$, $\forall x \in A$ and $\forall y \in B \ni f(A) \subseteq A$ and $f(B) \subseteq B$ with the condition that $f(A)$ is contained in a compact subset A_1 of A. Then $\{F_k^n(x)\}$ converges to a fixed point of f_i, $\forall x \in A \cup B$.

Proof. We can easily prove that the mappings F_j and $f_j F_{j-1}$, j = 1, 2, . . . , k. are relatively nonexpansive and map $A \cup B$ into itself.

Now we are going to prove $\{F_1, F_2, \ldots, F_k\}$ and $\{f_1, f_2, \ldots, f_k\}$ have the same set of common fixed points.

Let $x \in A \cup B$ with $f_j(x) = x$, j = 1, 2, . . . , k. Then

$$F_1(x) = (1 - \alpha)x + \alpha f_1 F_0(x) = (1 - \alpha)x + \alpha f_1(x) = (1 - \alpha)x + \alpha x = x,$$
$$F_2(x) = (1 - \alpha)x + \alpha f_2 F_1(x) = x$$

Proceeding like this, we get $F_j(x) = x$, j = 1, 2, . . . , k.

Now, let $F_j(x) = x$, j = 1, 2, . . . , k.

$$x = F_j(x) = (1 - \alpha)x + \alpha f_j F_{j-1}(x) = (1 - \alpha)x + \alpha f_j(x)$$
$$\Rightarrow \alpha x = \alpha f_j(x)$$

Hence $f_j(x) = x$, j = 1, 2, . . . , k.

Since (1) has the same form as (2), $\{F_k^n(x)\}$ conveges to a fixed point y of $f_k F_{k-1}$. We wish to show next that y is a common fixed point of f_k and $F_{k-1}(k \geq 2)$. To this we first show that $f_{k-1}F_{k-2}y = y$ $(k \geq 2)$. Suppose not, the closed line segment $[y, f_{k-1}F_{k-2}y]$ has positive length.

Let $z = F_{k-1}y = (1 - \alpha)y + \alpha f_{k-1}F_{k-2}(y)$

By hypothesis, there exists a point $w \in A \cup B$ such that $f_1 w = f_2 w = \cdots = f_k w = w$. Since f_i and F_i have the same common fixed points, it follows that $f_{k-1}F_{k-2}{-}w = w$.

By relatively nonexpansive, $\|f_{k-1}F_{k-2}y - w\| \leq \|y - w\|$ and $\|f_k z - w\| \leq \|z - w\|$ (3)

So w is atleast as close to $f_k z$ as to z.

But $f_k z = f_k F_{k-1} y = y$. Therefore w is atleast as close to y as to $z = (1 - \alpha)y + \alpha f_{k-1}F_{k-2}y$.

Since X is strictly convex, $\|y - w\| < \|f_{k-1}F_{k-2}y - w\|$, which is a contradiction to (3). Therefore $f_{k-1}F_{k-2}y = y$

Now, $F_{k-1} = (1 - \alpha)y + \alpha y = y$ and $y = f_k F_{k-1} y = f_k y$

$\Rightarrow$ y is a common fixed point of f_k and F_{k-1}. Repeating the argument, we conclude that y is a common fixed point of f_j and F_j, j = 1, 2, . . . , k

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